Borsuk Ulam in Combinatorics

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1 Intro

Lovasz' original proof of the Kneser conjecture was the inspiration. Meta idea: use the nonexistence of a map from the "configuration space" to the "target space". At this point introduce: B^n, S^n, continuous functions, open set, closed set.

2 Statement and equivalent forms

The main version of the theorem:

Theorem 2.1 (Borsuk Ulam v1). *For every map $f : S^n \to \mathbb{R}^n$, there is some $x \in S^n$ such that $f(x) = f(-x)$.*

Antipodal map: One that satisfies $f(-x) = -f(x)$.

Theorem 2.2 (Borsuk Ulam v2). *For every antipodal map $f : S^n \to S^{n-1}$, there is some x such that $f(x) = 0$.*

Theorem 2.3 (Borsuk Ulam v3). *There is no antipodal map from $S^n \to S^{n-1}$.*

Theorem 2.4 (Borsuk Ulam v4). *There is no continuous map $f : B^n \to S^{n-1}$ that is antipodal on the boudary.*

Theorem 2.5 (Borsuk Ulam vC). *For every cover of S^n by closed sets $F_1, \ldots, F_{n+1}$, there is some F_i with a pair of antipodal points.*

Theorem 2.6 (Borsuk Ulam vO). *For every cover of S^n by open sets $F_1, \ldots, F_{n+1}$, there is some F_i with a pair of antipodal points.*

Another important theorem:

Theorem 2.7 (Brouwer Fixed Point). *Every map $f : B^n \to B^n$ has a fixed point ($f(x) = x$).*

Important: Generalise the open/closed thing. The argument is by induction on closed sets For all open sets U_i, pick a lebesgue number λ for the covering, pick balls of radius λ at every point, find an open cover. Let F_i be the closure of the balls in the cover in U_i. These F_i are closed and cover, hence the theorem.

Suppose it holds for t closed sets and you have $t+1$. Fix some closed set F. If F has antipodes, done. If not, its diameter is $2 = \epsilon$. Let F' = all points at distance $\epsilon/2$ from F. This is open, and F' cannot contain antipodes by construction. We can apply the induction hypothesis now.

3 Kneser Conjecture

The Kneser graph $KG_{n,k}$ has vertex set $\binom{[n]}{k}$. Two vertices are adjacent if their intersection is empty. Conjecture: Chromatic number is $n-2k+2$ for $n \geqslant 2k-1$.

First, these many colours are enough: For each vertex v, assign it the colour defined by $\min((\min i : i \in v), n-2k+2)$. If two vertices get the same colour less than $n-2k+2$, that element is common so no edge. If they are equal to $n-2k+2$, then all elements lie in $\{n-2k+2, n-2k+3, \ldots, n\}$ and there are only $2k-1$ such, so they intersect.

This is tight: Let $d := n-2k+1$. Pick a set of points $X \subset S^d$ in general position: no hyperplane through the center has more than d points. This is possible.

Let the vertex set of the graph be $\binom{X}{k}$. Suppose there is a colouring with d colours. Define sets $A_1, \ldots, A_d \subseteq S^d$ as follows. For an $x \in S^d$, look at the open half sphere centered at x. Consider all k tuples of elements in this half sphere. If there is a tuple with colour i, then $x \in A_i$. Let $A^{d+1} = S^d \setminus \cup A_i$.

The sets $A_1, \ldots A_d$ are open, last is closed. By borsuk ulam, antipodal $y, -y$ points in some A_i. Suppose in A_{d+1}. That $y \in A_{d+1}$ implies that in its halfsphere, there are atmost $k-1$ points of X. That $-y \in A_{d+1}$ implies that in its halfsphere, atmost $k-1$ points of X. Therefore, there must be atleast $n-2k+2 = d+1$ points in the "equator" of these two halfspheres. This contradicts the general position assumption. Suppose then $y, -y \in A_j$ with $j \leqslant d$. Then there is some k tuple of X with colour j in the halfsphere centered at y, and another tuple with colour j in the halfsphere centered at $-y$. These are disjoint, so the tuples are disjoint, so there is an edge between them, contradicts colouring.

4 Ham Sandwich

Given d objects in R^d with positive "volume" there is a hyperplane that divides every set in two.

Proof. Let $A_1, \ldots, A_d$ be the sets. Consider the sphere S^d, and a point $u = (u_0, \ldots, u_d)$. If some $u_1, \ldots, u_d$ is nonzero, then assign to u the halfspace in R^d defined by $u_1x_1 + u_2x_2 +$

$\cdots + u_d x_d \leqslant u_0$. To $(1, 0, \ldots, 0)$ just assign the halfspace than is the whole of $\mathbb{R}^d$, and to the opposite point assign the space ϕ.

Define function $f : S^d \to \mathbb{R}^d$ as sending the point u to the vector of volumes of each A_i in the halfspace corresponding to u. We must have $f(y) = f(-y)$ for some y. Note that y cannot be $(1, 0, \ldots, 0)$ since we know the function is not equal to that at the opposite point, so it must be a halfplane.

If we have more objects, use a hypersurface: a degree k hypersurface can bisect $\binom{k+n}{n} - 1$ many objects. □

More interesting, Ham sandwich for point sets. Let $A_1, \ldots, A_d \subset \mathbb{R}^d$ be finite point sets. Then there is a hyperplane that bisects them Here, bisects means that on each side there are atmost $\lfloor |A_i|/2 \rfloor$ points. If odd number of points say $2k+1$, then each side has atmost k points and atleast 1 on the hyerplane.

Proof. Idea is to replace each point in A_i by tiny balls.

First assume everything is in general position (no $d+1$ on a hyerplane) and no common points and all A_i are odd sized. Pick a small enough ϵ so balls of radius ϵ around each point do not intersect. Pick a bisecting hyperplane. This plane has to bisect atleast one ball from each A_i since there are odd number. If it bisects more than 1 from some A_i then more than $d+1$ points on the hyerplane contradicting general position. So exactly one from each.

Next suppose odd cardinality but not general position. Then perturbation argument. For each η, perturb every point by atmost η so that they are in general position and pick the bisecting hyperplane h_η. Each hyperplane is a point in $\mathbb{R}^{d+1}$, given by $\langle a, x \rangle = b$. If we normalise so that a is always a unit vector, then the set of hyerplanes h_η is a bounded set, since for each of them b_η is bounded by say the diameter of all the A_i. So there is a converging subsequence that converses to point (p, q) or hyperplane $\langle p, x \rangle = q$. Consider sequence $\eta_1 \geqslant \eta_2 \geqslant \cdots$ that converges to 0 with hyerplanes converging to (p, q), call this hyperplane h. For a particular x, if it is one on side of h, say $\langle p, x \rangle > q$, then it is at distance δ from h. Then there is some η_j such that it is at distance $\delta/2$ from h_{η_j} and from all $h_{\eta_{j'}}$ with $j' > j$. So if there are k points from A_i on one side of h, then starting from some η_j with j large enough, these k points are on one side of h_{η_j}. Since the h_{η_j} bisect, it must be that $k \leqslant \lfloor |A_i|/2 \rfloor$ and thus the h also bisect.

Finally, if there are even number of points, then just delete a point and make it odd and find h. If we add the point back, h will still bisect by definition. □

We can als get the following nice version: If the points are in general position, then there is a hyerplane than bisects with exactly $\lfloor |A_i|/2 \rfloor$ on each side, and atmost 1 from each A_i on the hyerplane.

5 Necklace Theorem

Simple consequence of the discrete ham sandwich.

Theorem 5.1 (Akiyama and Alon)**.** *Let* $A_1, \ldots A_d \subset \mathbb{R}^d$ *having* n *points each that are in general position. Think of each point in the union as being coloured by the colour* i*. Then we can partition the union into* n *many* d *tuples, with each tuple consisting of one point from each* A_i*, such that the convex hull of each tuple is disjoint.*

In two dims: pick a set of n red and n blue points in the plane. You can match pairs of red and blue so that the lines joining the pairs do not intersect.

Proof. Induct on n. For $n > 1$ odd, pick hyerplane that bisects each A_i and that has has exactly one point of each colour. There is one d tuple on h, and then invoke the theorem on each side. □

Necklace division. Open necklace, d types of stones, even number of each kind. How many cuts to divide. Easy to see d is neccessary: put all stones of type 1 first, then type 2 and so on. Also d is sufficient.

Proof. Note than the moment curve is a curve in $\mathbb{R}^d$ parametrised by $t \to (t, t^2, \ldots, t^d)$. Call this function $\gamma(t)$. Property: Every hyerplane intersects the curve at atmost d points, and if it intersects at exactly d points then the curve crosses the hyerplane at each point of intersection.

Place the necklace along the moment curve. Let A_i be the set of all stones of type i. So we have

$$A_i = \{\gamma(k) | \text{the stone in position k is of type i}\}$$

By ham sandwich, there is a hyerplane bisecting. Since the type of each stone is even, the hyerplane itself contains no stones. Further it bisects at atmost d points on the curve. These are the required cuts. □